Geometric characteristics of conics in Bézier form

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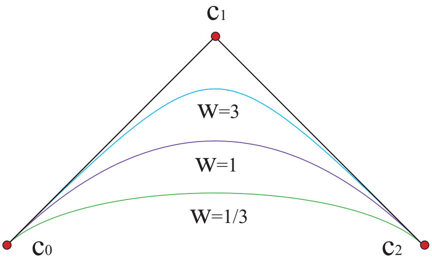
Aim

- Derive coordinate-free expressions for geometric characteristics of conics written in Bézier form in terms of their control points and weights.

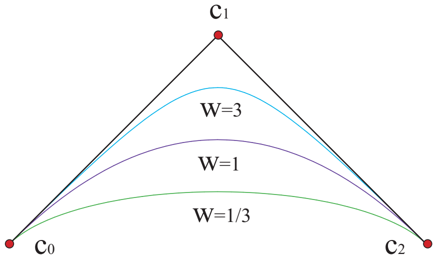


Previous work

- E.T.Y. Lee, in: G. Farin (Ed.), *Geometric Modeling: Algorithms and new trends*, SIAM, 3–20 (1987).
- G. Albrecht, in: T. Lyche, L. Schumaker (Eds.), *Mathematical methods for curves and surfaces, Oslo 2000*, Vanderbilt University Press, 15–24 (2001).
- R. Goldman, W. Wang, *International Journal of Computational Geometry and Applications* **14**, 161 – 187 (2004).
- J. Sánchez-Reyes, *Computer Aided Geometric Design* **21**, 111 – 116 (2004).
- C. Xu, T-w. Kim, G. Farin, *Computer Aided Geometric Design* **27**, 458 – 460 (2010).
- J. Sánchez-Reyes, *Computer Aided Geometric Design* **28**, 345 – 348 (2011).

New approach

- Coordinate-free formulas for geometric characteristics.
- Use affine and projective geometry.
- Why? Some features (axes, vertices, eccentricity. . .) are Euclidean, but other ones (center, asymptotes, foci. . .) are not necessarily so.
- Key point: obtaining a coordinate-free expression for the implicit equation of the conics.

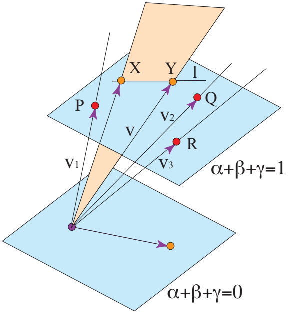


Outline

Projective plane

- The projective plane is the set of vector lines of $\mathbb{R}^3$.
- Each *point* X of the plane is characterised by a vector $\vec{v}$ (or $\lambda\vec{v}$).
- A *line* l in the projective plane is a vector plane of $\mathbb{R}^3$, $l(X) = 0$.
- A *frame* of the plane is a set $\{P, Q, R\}$ of non-alligned points.
- Choosing $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ along them, we may write any point X as

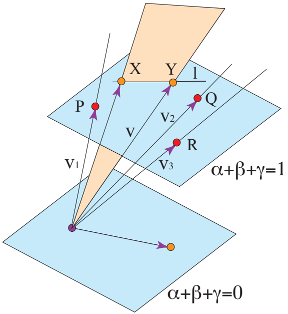
$$X = \alpha P + \beta Q + \gamma R, \tag{1}$$



Points and vectors

Once a frame $\{P, Q, R\}$ is chosen, points of the projective plane are divided in two types:

- Points such that $\alpha + \beta + \gamma \neq 0$. These are the points of the affine plane.
- Points such that $\alpha + \beta + \gamma = 0$. These points of the projective plane form the *line at infinity*. They are the directions of vectors of the affine plane.



Bézier conic arcs

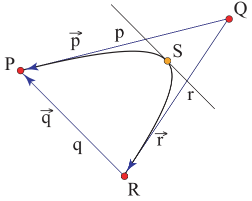
We consider conic arcs in Bézier form,

$$c(t) = \frac{c_0(1-t)^2 + 2wc_1t(1-t) + c_2t^2}{(1-t)^2 + 2wt(1-t) + t^2}, \quad t \in [0,1],$$

defined by control polygon, $\{c_0, c_1, c_2\}$, and weights, $\{1, w, 1\}$

- We denote $P = c_0$, $Q = c_1$, $R = c_2$.
- We denote by p, q, r the lines of the edges of the polygon.
- The line p is tangent the conic at P and r is tangent at R.
- We fix the linear maps p, q, r requiring that

$$p(R) = 1, \quad q(Q) = 1, \quad r(P) = 1.$$



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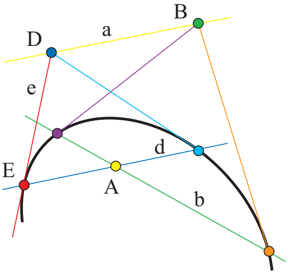
- We denote $P = c_0$, $Q = c_1$, $R = c_2$.
- We denote by p, q, r the lines of the edges of the polygon.
- The line p is tangent the conic at P and r is tangent at R.
- We take $\{P, Q, R\}$ as our frame for the projective plane.
- $\{p, q, r\}$ is the frame for the dual projective plane (lines on the projective plane). For instance, the line at infinity is $p + q + r$,

$$0 = (p+q+r)(\alpha P + \beta Q + \gamma R) = \alpha + \beta + \gamma.$$

Polar lines

For a conic section the *polar line* a of a point A is defined as:

- If A lies on the conic, the polar line a of A is the tangent line at A.
- If A lies out of the conic, we trace tangents to the conic through A. They meet the conic at two points. The line linking them is the polar line a of A.
- If we cannot draw the tangents from A, we may obtain its polar line a by linking two points B and D with polar lines b, d meeting at A.



Notation: We denote by p the polar line of a point P.

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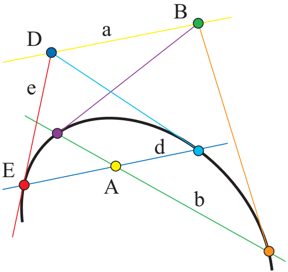
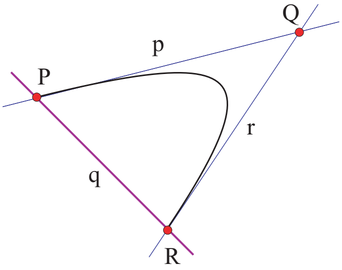


If $C(X, X) = 0$ is the equation of the conic, then $p(X) = C(P, X)$.

Conic equations

- A conic is determined by 5 points.
- We have 4 points already: Double points P, R.
- There are two degenerate conics through them: The double line $q \cup q$ and the intersecting lines $p \cup r$.
- The set of conics through them is the pencil of conics $pr + \lambda q^2$,

$$0 = p(X)r(X) + \lambda q(X)^2.$$



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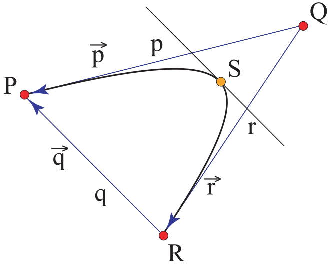
$$0 = p(X)r(X) - \frac{1}{4w^2}q(X)^2.$$



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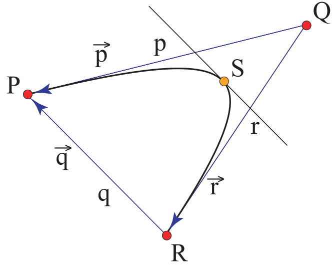
- The main advantage of this expression is that it is coordinate free.

(pencil.mov)

Tangential conic equations

- We may view a conic as the set of its tangent lines.
- We know that p and r are the tangents to the conic at P and R (4 conditions).
- Two degenerate conics $P \cup R$ and $Q \cup Q$ fulfill such conditions.
- And define the pencil of tangential conics $PR + \mu Q^2$.
- We fix μ using the tangent line s to the shoulder point S.
- A line x is tangent to the conic if

$$x(P)x(R) - w^2 x(Q)x(Q) = 0.$$



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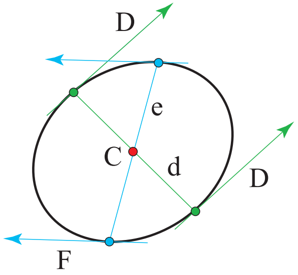
- This is useful since we avoid inversion of C in $p(X) = C(P, X)$ for obtaining a point P, given its polar line p.

Centers and diameters

- The *diameters* of the conic are the polar lines of the points on the line at infinity, since the tangents to the conic at the intersections with a diameter are parallel.
- The *center* is the intersection of all diameters.
- Hence, the polar line of the center is the line at infinity.



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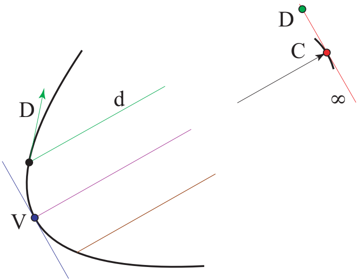
- Hence, the polar line of the center is the line at infinity.
- Using the tangential equation of the conic we get

$$C = \frac{P + R - 2w^2 Q}{2 - 2w^2}, \qquad w \neq 1.$$

Parabola

- The center of a parabola is a direction, not a point, $C = \vec{p} + \vec{r}$. The parabola is tangent to the line at infinity at its center.
- All diameters are parallel and have the direction of the center.
- The direction of the tangent to the intersection of a diameter d and the conic is given by D.
- The *axis* is a diameter for which D and C are orthogonal
- The *vertex* is the intersection of the axis with the conic



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$$2\langle\vec{r},\vec{p}+\vec{r}\rangle p(X) + \left(\|\vec{r}\|^2 - \|\vec{p}\|^2\right) q(X) - 2\langle\vec{p},\vec{p}+\vec{r}\rangle r(X) = 0.$$

- The *vertex* is the intersection of the axis with the conic

$$V = \frac{\langle\vec{r},\vec{p}+\vec{r}\rangle^2 P + 2\langle\vec{p},\vec{p}+\vec{r}\rangle\langle\vec{r},\vec{p}+\vec{r}\rangle Q + \langle\vec{p},\vec{p}+\vec{r}\rangle^2 R}{\|\vec{p}+\vec{r}\|^4}.$$

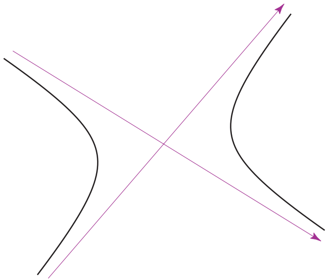
Asymptotes

- A conic meets the line at infinity at two *points* (real or complex), the *asymptotic directions*,

$$\vec{v}_\pm = \left(w \pm \sqrt{w^2-1}\right)\vec{p} + \left(w \mp \sqrt{w^2-1}\right)\vec{r}.$$

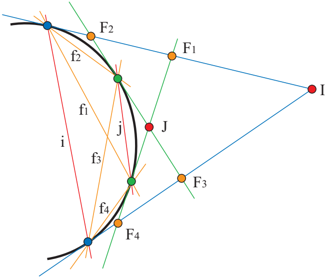
- The asymptotes are the tangents or polar lines to such points at infinity,

$$\left(w \pm \sqrt{w^2-1}\right)p(X) + \left(w \mp \sqrt{w^2-1}\right)r(X) + \frac{q(X)}{w} = 0.$$



Foci

- There are two complex points, I, J, at the line at infinity where all circles meet, *absolute or circular points*.
- We draw the four tangents to the conic from them.
- These tangents meet at the *foci* (2 real + 2 complex) of the conic.
- Their polar lines are the directrices of the conic.
- We compute the intersections of the polar lines of I, J with the conic: The four directrices are the lines linking them.



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- We compute the intersections of the polar lines of I, J with the conic: The four directrices are the lines linking them.
- The equations of the real directrices are

$$w|\alpha_\pm|^2 p(X) - \Re(\alpha_\pm) q(X) + w r(X) = 0,$$

$$\alpha_\pm = \frac{b \pm \sqrt{b^2 - 4w^2 ac}}{2wc}, \quad I = aP + bQ + cR.$$

- And the real foci are

$$F_\pm = \frac{|\alpha_\pm|^2 P + 2w\Re(\alpha_\pm) Q + R}{|\alpha_\pm|^2 + 2w\Re(\alpha_\pm) + 1}.$$

Focus of a parabola

- The polar lines of the absolute points are diameters and meet at the center, which is a point of the parabola.
- There are just three intersections of the polar lines of I, J with the parabola: One is the center. The other two define the only directrix,

$$\|\vec{r}\|^2 p(X) - \langle \vec{p}, \vec{r} \rangle q(X) + \|\vec{p}\|^2 r(X) = 0.$$

- It is the polar line of the only focus,

$$F = \frac{\|\vec{r}\|^2 P + 2\langle \vec{p}, \vec{r} \rangle Q + \|\vec{p}\|^2 R}{\|\vec{p} + \vec{r}\|^2}.$$

Further characteristics

- The eccentricity, the axes, the vertices of a conic arc may also be computed, but the expressions are lengthier.
- They require Euclidean geometry.

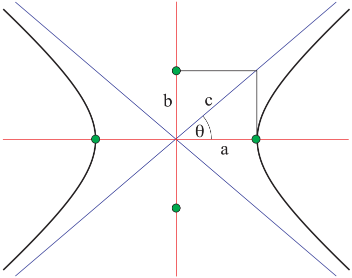
Eccentricity

- The *eccentricity* of a conic is defined as the quotient $e = c/a$.
- The angle between asymptotes may be computed from

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{a^2 - b^2}{c^2} = \frac{2}{e^2} - 1.$$

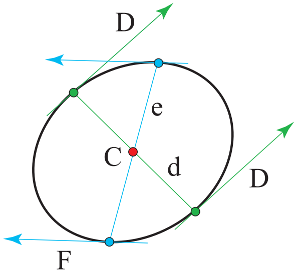
- From it we get the eccentricity, taking into account that we already know the asymptotic directions $v_\pm$,

$$\langle v_+, v_-\rangle = \|\vec{p} - \vec{r}\|^2 + 4w^2\langle \vec{p}, \vec{r}\rangle \ldots$$



Axes

- An *axis* is a diameter which is orthogonal to the tangent lines at the intersection points with the conic.
- The diameters $d = \pi p + \rho q + \sigma r$ satisfy $\pi + \sigma - 2w^2\rho = 0$.
- And their direction is $\vec{v} = (\rho - \pi)\vec{p} + (\sigma - \rho)\vec{r}$.
- A diameter d is the polar line of $D = \pi\vec{p} + \sigma\vec{r}$, direction of the tangents to the conic at the intersections with the diameter.
- For obtaining the direction of the axes, we require that $\langle D, \vec{v} \rangle = 0$.



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- For obtaining the direction of the axes, we require that $\langle D, \vec{v} \rangle = 0$.
- The equation of the axes has

$$\pi = w^2\rho + 1, \qquad \sigma = w^2\rho - 1,$$

$$w^2(w^2 - 1)\rho^2 + \frac{2w^2\left(\|\vec{r}\|^2 + \|\vec{p}\|^2\right) - \|\vec{q}\|^2}{\left(\|\vec{p}\|^2 - \|\vec{r}\|^2\right)}\rho + 1 = 0.$$

Conclusions

- We have obtained coordinate-free equations for conic arcs in terms of their control polygons and weights, both in point and tangential form, making use of projective geometry.
- We have obtained closed formulas for geometric characteristics of conics in a coordinate-free fashion using those equations.
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